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## Vortices on ‘rail road tracks’: a possible realization of a random matrix ensemble

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**Abstract.** The thermodynamics of interacting vortices constricted to move in one-dimensional tracks carved out on superconducting films in the extreme type II limit is mapped into the Coulomb gas model of random matrices. Application of the Selberg integral supplies *exact* expressions for the constitutive relation for two configurations of tracks.

In the present paper we derive an *exact* expression for the constitutive relation between the external magnetic field,  $H$ , and the magnetic induction,  $B$ , inside a type II superconductor, i.e., we determine the relationship  $B = B(H)$ . The induction inside the superconductor is determined by the density,  $n$ , of magnetic vortices. Each vortex carries one flux quantum  $\Phi_0$ , hence  $B = \Phi_0 n$  [1]. In order to be able to find an exact expression for  $B(H)$  we restrict ourselves to thin superconducting films. The interaction energy of two vortices depends in this case logarithmically on the distance of separation of the two vortices. Furthermore, we assume that the vortices are confined to move on tracks engraved into the superconducting film, see figure 1. This experimental situation has in fact been studied by Pruymboom *et al* [2]. The thickness of the film is reduced in the track. Since there is a loss of condensation energy in the core of a vortex, it will be energetically favourable to position a vortex with the core on the track rather than outside the track, where the core would have to drill its way through a thicker layer of superfluid.

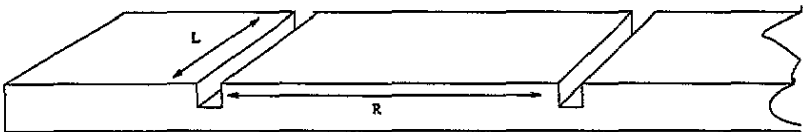


Figure 1. Sketch of a superconducting thin film with tracks engraved into the surface.

The combined simplification of logarithmic interaction and restriction on the configurational degrees of freedom allow us to map the problem onto the Coulomb gas model of random matrices [3]. The partition function is known exactly for this model and the constitutive relation readily derived.

Let us now describe the derivation of the relation between  $H$  and  $B$ . The partition function for a fixed number of  $N$  two-dimensional vortices on a track is given by

$$Z_N = \left( \prod_{a=1}^N \int_0^L \frac{dx_a}{l} \right) \exp(-\beta E[x_1, \dots, x_N]) \quad (1)$$

where

$$E[x_1, \dots, x_N] = NE_c - E_1 \sum_{1 \leq a < b \leq N} \ln \left( \frac{|x_a - x_b|}{\lambda_{\text{eff}}} \right) \tag{2}$$

where  $E_c (> 0)$  is the energy of the vortex core and  $E_1 (> 0)$  sets the scale of the vortex interaction [4]. Both  $E_c$  and  $E_1$  are temperature dependent [4]. The length of the tracks is denoted by  $L$ , see figure 1. The scale of the logarithmic interaction is set by the effective magnetic penetration depth,  $\lambda_{\text{eff}}$ , of the film. Expressed in terms of the bulk penetration depth  $\lambda$  and the film thickness  $d$  one has  $\lambda_{\text{eff}} = \lambda^2/2d$  [5].  $\beta = 1/T$  is the inverse temperature.  $N$  is related to the magnetic induction  $B$  through  $BLR = N\Phi_0$ , where  $R$  is the distance between the individual tracks, see figure 1.  $l$  is the length scale of the configuration space and may be taken to be equal to  $\xi(0)$ , the coherence length at zero temperature. We assume that  $L < \lambda_{\text{eff}}$  and  $R \gg \lambda_{\text{eff}}$ .

The condition  $\lambda_{\text{eff}} \gg L$  ensures that the logarithmic interaction is valid for all possible separations of a pair of vortices on the same track. Beyond the distance  $\lambda_{\text{eff}}$  the vortex interaction decays exponentially due to the diamagnetic screening currents [1, 5]. Consequently, we can neglect the interaction between vortices on different tracks if the condition  $R \gg \lambda_{\text{eff}}$  is fulfilled.

It should be noticed that the vortices will stay in the tracks for magnetic fields less than a certain field strength  $H^*$ . At  $H^*$  the repulsion between the flux lines becomes so strong that it will be able to overcome the pinning force exerted by the carved track. The actual value of  $H^*$  will depend on the depth of the groove and in a first approximation will be equal to the  $H_{c1}$  of the bulk material. The partition function in equation (1) applies for fields  $H < H^*$ .

The partition function can be re-written as

$$Z_N = e^{-N\bar{E}_c} \left( \frac{L}{l} \right)^N \left( \frac{L}{\lambda_{\text{eff}}} \right)^{N(N-1)\bar{E}_1} \left( \prod_{a=1}^N \int_0^1 dt_a \right) \prod_{1 \leq a < b \leq N} |t_a - t_b|^{2\bar{E}_1} \tag{3}$$

where we have introduced the following dimensionless energies:

$$\bar{E}_c = \frac{E_c}{T} \quad \bar{E}_1 = \frac{E_1}{2T} \tag{4}$$

and the dimensionless integration variables  $t_a$ . The multiple integral in equation (3) is a special case of the Selberg integral [6]; with its aid we obtain the partition function and the free energy of this system:

$$-\frac{F_N}{T} = \ln Z_N = N [\ln(L/l) - \ln \Gamma(1 + \bar{E}_1) - \bar{E}_c] + N(N-1)\bar{E}_1 \ln(L/\lambda_{\text{eff}}) + \sum_{j=0}^{N-1} \ln \left[ \frac{\Gamma(1 + (j+1)\bar{E}_1)\Gamma^2(1 + j\bar{E}_1)}{\Gamma(2 + (N-1+j)\bar{E}_1)} \right]. \tag{5}$$

An expression for the lower critical field  $H_{c1}$  is obtained from the condition that the Gibbs free energy of the state with a single vortex  $F_1 - H_{c1}BV/4\pi$  ( $V = LRd$  is the volume per track) is equal to the Gibbs free energy of the state without any vortices  $F_0 = 0$  [1]. The result is

$$H_{c1} = \frac{4\pi}{\Phi_0 d} (E_c - T \ln(L/l)). \tag{6}$$

The constitutive relation is obtained in the following way which makes use of the fact that  $H$  acts as the chemical potential for the vortices and  $dG/dN = 0$ , where  $G$  is the Gibbs free energy for the system [7]:

$$\frac{\partial F_N/T}{\partial N} = \frac{1}{4\pi} \frac{\Phi_0 H}{LR T} V. \tag{7}$$

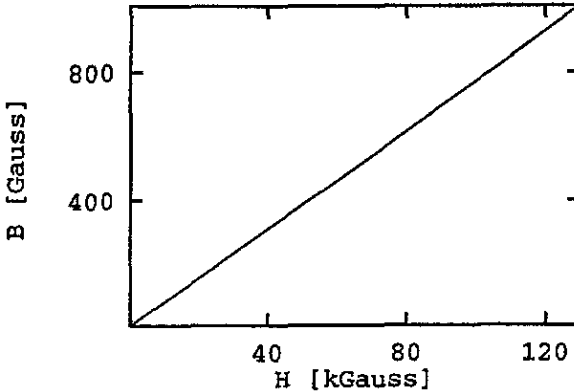


Figure 2. The induction  $B$  as a function of the external magnetic field  $H$  obtained from equation (8) for the following set of parameters:  $L = 1 \mu\text{m} = R/10$ ,  $d = 10 \text{ \AA}$ , and  $\lambda = 1500 \text{ \AA}$ . The temperature is taken as 10 K.

We simplify the above expression in the large- $N$  limit, by replacing the sum of the logarithms by an integral while discarding the differences between  $N$  and  $N \pm 1$ . With the aid of  $\Gamma(ax + b) \simeq (2\pi)^{1/2} e^{-ax} (ax)^{ax+b-1/2}$ ,  $a > 0$ ,  $b > 0$ ,  $x \gg 1$  we find

$$\frac{dH}{4\pi T} \Phi_0 = 2X \ln(4\lambda_{\text{eff}}/L) + \ln X \tag{8}$$

where  $X := N \bar{E}_1 = (BLR/\Phi_0) \bar{E}_1$ , and we have discarded the  $N$ -independent and  $O(N^{-1})$  terms. The magnetic permeability  $\mu$ , in the high-field limit, can be found by further discarding the  $\ln X$  term

$$B = \mu_0 H \tag{9}$$

with

$$\mu_0 = \frac{d\Phi_0^2}{4\pi E_1 LR \ln(4\lambda_{\text{eff}}/L)}. \tag{10}$$

The constitutive relation has a weak logarithmic correction for not too large  $H$ . This is found by the substitution  $X = X_0 + \delta X$ , where

$$\frac{\delta X}{X_0} = -\ln \left( \frac{dH \Phi_0 / [8\pi T \ln(4\lambda_{\text{eff}}/L)]}{1 + dH \Phi_0 / 4\pi T} \right) \tag{11}$$

which in turns supplies a non-linear permeability  $\mu(H) = \mu_0(1 + \delta X/X_0)$ . The permeability is slightly reduced from high-field values which is to be expected.

So far we have considered the ensemble where the positions of the vortices are restricted to a straight line of length  $L$ . Let us now consider a system in which the vortices are confined to a circular track of radius  $R_0$ . In this case we avoid edge effects. The thermodynamics

of the logarithmically interacting vortices on a ring are obtained from Dyson's circular ensemble [8]. The interaction energy of the vortices is given by

$$E^c[\theta_1 \dots \theta_N] = NE_c - E_1 \sum_{1 \leq a < b \leq N} \ln \left( \frac{R_0}{\lambda_{\text{eff}}} |e^{i\theta_a} - e^{i\theta_b}| \right) \quad (12)$$

and the partition function

$$Z_N^c = \left( \prod_{a=1}^N \int_0^{2\pi} \frac{R_0}{2\pi l} d\theta_a \right) \exp(-\beta E^c[\theta_1 \dots \theta_N]) \quad (13)$$

where  $R_0$  ( $\ll \lambda_{\text{eff}}$ ) is the circumference of the circle. The constitutive relation is in this case easily obtained directly from the difference  $F_{N+1} - F_N$  in free energy. We have

$$-\left( \frac{F_{N+1}^c - F_N^c}{T} \right) = 2N \bar{E}_1 \ln(R_0/\lambda_{\text{eff}}) + \ln \left[ \frac{\Gamma((N+1)\bar{E}_1)}{\Gamma(N\bar{E}_1)} \right]. \quad (14)$$

For large  $N$  and with the aid of the large- $x$  behaviour of  $\Gamma(ax+b)$ , we find

$$\frac{\Phi_0 H}{8\pi^2 R_0 d T} V = 2 \ln(\lambda_{\text{eff}}/2\pi R_0) Y + \bar{E}_1 \ln Y \quad (15)$$

where

$$Y := N \bar{E}_1. \quad (16)$$

The result for the permeability in the high-field limit reads

$$\mu_0^c = \frac{1}{32\pi^2 E_1 R_0} \frac{\Phi_0^2}{\ln(\lambda_{\text{eff}}/2\pi R_0)} \quad (17)$$

a result similar to that of the straight track.

We have derive an exact expression for the constitutive relation for an artificially patterned superconducting thin film. The result should be directly experimentally accessible. Pinning along the the engraved track may in principle influence the motion of the vortices, although it appears that for deep grooves pinning should affect the motion parallel to the track rather insignificantly.

Our future extension of the present calculation will be to apply random matrix theory to the dynamics of the vortices in the constricted geometry.

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